Journal of Statistical Physics, Vol. 75, Nos. 1/2, 1994

Nonuniversality and Continuity of the Critical Covered Volume Fraction in Continuum Percolation

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Received September 24, 1993; final January 3, 1994

We establish, using mathematically rigorous methods, that the critical covered volume fraction (CVF) for a continuum percolation model with overlapping balls of random sizes is not a universal constant independent of the distribution of the size of the balls. In addition, we show that the critical CVF is a continuous function of the distribution of the radius random variable, in the sense that if a sequence of random variables converges weakly to some random variable, then the critical CVF based on these random variables converges to the critical CVF of the limiting random variable.

KEY WORDS: Poisson point process; continuum percolation; critical intensity; covered volume fraction.

1. INTRODUCTION AND RESULTS

In the literature on continuum percolation, two related parameters have been studied. The first is the *covered volume fraction* (CVF), which has been studied primarily by physicists (Scher and Zallen,[10] Pike and Seager,[8] Kersetz and Vicsek,[4] Gawlinski and Redner,[1] Phani and Dhar[7]), while the other is the *intensity* of the underlying point process, studied primarily by mathematicians (Hall,[2] Menshikov,[6] Roy[9]). The results obtained in the first set of work is limited in that the results are primarily based on Monte Carlo simulations, while the latter set of work is limited in that the results primarily pertain to the existence of the percolating regime in a setting where the balls are random but of a given fixed distribution. In this paper we shall be concerned with two aspects of

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the CVF at criticality. First, we settle a question raised in the first set of work regarding the universality of the critical CVF by methods established in the second set of work. Second, we obtain a continuity result concerning the critical CVF when the radii converge weakly.

The model of continuum percolation consists of overlapping d-dimensional balls each of which are of random radius and are centred in a 'uniform manner' on $\mathbb{R}^d$; more precisely, for every $i = 1, 2,\ldots$, each point x_i of a Poisson point process X of intensity λ on $\mathbb{R}^d$ in the center of a ball $S(x_i, \rho_i)$ of radius ρ_i, where $\rho_1, \rho_2,\ldots$ is an independent and identically distributed sequence of random variables which are all independent of the underlying Poisson process. Let ρ denote a random variable whose distribution is independent of X and $\{\rho_i, i \geqslant 1\}$ and is identical to that of ρ_1. We denote this model by the triple (X, λ, ρ). The probability measure governing this process will be denoted by $P_{\lambda,\rho}$ and $E_{\lambda,\rho}$ is the corresponding expectation operator. The part of the space which is covered by at least one ball will be denoted by C and the uncovered (vacant) part by V.

In a realization of this model let $x_1,\ldots, x_n$ be all the points in the unit box $[0, 1]^d$ and $r_1,\ldots, r_n$ the associated radii of these balls at these points. Consider the quantity $\sum_{1 \leqslant i \leqslant n} \pi_d r_i^d$, where π_d denotes the d-dimensional volume of a ball of unit radius. This corresponds to the sum of the volumes of each of the balls centered in the box $[0, 1]^d$. It can be easily seen that the expected sum of the volumes of each of the balls centered in the unit box $[0, 1]^d$ is $\lambda \pi_d E_{\lambda,\rho} \rho^d$. This quantity is called the *volume density* of (X, λ, ρ). By the invariance properties of the model it is obvious that the volume density is unaffected if instead of $[0, 1]^d$ we chose a different unit box in $\mathbb{R}^d$. The CVF is the quantity $1 - \exp(-\lambda \pi_d E_{\lambda,\rho} \rho^d)$, which corresponds to the expected volume in a unit box covered by balls (see Hall,[3] p. 128). A simple argument using the ergodic theorem yields that if B_n is the box $[-n, n]^d$, the limit

$$\lim_{n \to \infty} \frac{1}{(2n)^d} \operatorname{vol}(B_n \cap C)$$

exists and is equal to $1 - \exp(-\lambda \pi_d E^d_{\lambda,\rho} \rho^d) = \text{CVF}$, where $\operatorname{vol}(A)$ denotes the d-dimensional volume of a region $A \subseteq \mathbb{R}^d$.

Note that if the radius random variable ρ is such that $E_{\lambda,\rho} \rho^d = \infty$, then for every $\lambda > 0$, the CVF equals 1. As such the whole space is covered in this case. To rule out such instances and to be able apply the known mathematical results of this model we restrict ourselves to the case where ρ has bounded support.

For $x \in \mathbb{R}^d$, let $C(x)$, the *cluster* of x, denote the connected component of C which contains x. The cluster of the origin is denoted by $C(0)$ and for $A \subseteq \mathbb{R}^d$, $C(A)$ denotes the union of all components in C which intersect A. Clearly, for a fixed ρ, if $\lambda_1 \leqslant \lambda_2$ then

$$P_{\lambda_1, \rho}(C(0) \text{ is unbounded}) \leqslant P_{\lambda_2, \rho}(C(0) \text{ is unbounded})$$

This allows us to define the *critical intensity* as

$$\lambda_c(\rho) := \inf\{\lambda: P_{\lambda, \rho}(C(0) \text{ is unbounded}) > 0\}$$

The *critical volume density* and the *critical CVF* are defined as $\lambda_c \pi_d E_{\lambda_c, \rho} \rho^d$ and $A_c(\rho) := 1 - \exp(-\lambda_c \pi_d E_{\lambda_c, \rho} \rho^d)$, respectively.

It is obvious that if ρ_1 and ρ_2 are such that $\rho_1 \equiv r_1$ and $\rho_2 \equiv r_2$ for some fixed reals $0 < r_1 < r_2 < \infty$, we have

$$\lambda_c(r_1) := \lambda_c(\rho_1) \geqslant \lambda_c(\rho_2) =: \lambda_c(r_2)$$

In addition, a simple rescaling argument (Zuev and Sidorenko[11]) yields

$$\lambda_c(r_1) r_1^d = \lambda_c(r_2) r_2^d \tag{1.1}$$

Clearly, (1.1) implies that

$$A_c(r_1) := A_c(\rho_1) = A_c(\rho_2) =: A_c(r_2) =: A_c \quad \text{(say)} \tag{1.2}$$

The equality in (1.2) suggested the conjecture (Kersetz and Vicsek[4]) that, for all random variables ρ with bounded support, $A_c(\rho)$ is a constant independent of ρ. Phani and Dhar[7] gave a heuristic argument which showed that the conjecture is false, and supported their argument with Monte Carlo simulations.

In this paper we prove the following:

Theorem 1.1. There exists a random variable ρ taking values a and b with probability p and $1-p$, respectively, where $a \neq b$, $a > 0$, $b > 0$, and $0 < p < 1$ such that

$$A_c(\rho) > A_c \tag{1.3}$$

where, as in (1.2), A_c denotes the critical CVF of a model with balls of fixed radius.

Our second result is concerned with the continuity of $A_c(\rho_k)$ when the sequence $\{\rho_k\}$ converges weakly:

Theorem 1.2. Let ρ_k and ρ be random variables such that for some

$R>0$ we have $0 \leqslant \rho \leqslant R$ and $0 \leqslant \rho_k \leqslant R$ a.s. for all $k \geqslant 1$. If $\rho_k \Rightarrow \rho$, then $A_c(\rho_k) \to A_c(\rho)$, where '$\Rightarrow$' denotes weak convergence.

In combination with Theorem 1.1, this result shows that for a whole class of distributions of ρ, inequality (1.3) is valid. Also, it states that simulation methods cannot distinguish between two models very 'close' to each other. In Section 3 we obtain further results on bounds on the rate of convergence of the critical intensities.

Our proof of Theorem 1.1 in Section 2 strongly suggests that whenever ρ is not a constant a.s., then $A_c(\rho) > A_c$. However, we do not have a proof of this inequality.

Finally, denote the critical volume density of a model with fixed-size balls by VD_c. Our proof of Theorem 1.1 also shows that for any $\varepsilon > 0$, it is possible to construct a model such that the critical volume density of this model is between $2VD_c - \varepsilon$ and $2VD_c$. This justifies a claim made by Phani and Dhar.[7]

For ease of description we present the proofs in the two-dimensional case; all our arguments, however, are valid in a higher-dimensional setting.

2. PROOF OF THEOREM 1.1

Let $0 < r_1 < r_2 < \infty$ be arbitrary positive numbers. Fix, $\varepsilon, \delta > 0$ such that

$$(2-\varepsilon-\delta)A_c - (1-\varepsilon)(1-\delta)A_c^2 > A_c \tag{2.1}$$

The expression in (2.1) will become clear in a moment. Next we choose $\lambda_2 < \lambda_c(r_2)$ such that the CVF of (X, λ_2, r_2) is equal to $(1-\varepsilon)A_c$. Also choose $\lambda_1 < \lambda_c(r_1)$ such that the CVF of $(X, \lambda_1, r_1) = (1-\delta)A_c$. Note that both processes are subcritical. Next we consider the superposition of these processes. We claim that the CVF of this superposition is strictly larger than A_c. To see this, note that it follows from Fubini's theorem and the ergodic theorem that the CVF of a process is equal to the probability that the origin (or any other point, for that matter) is covered. But by independence, the probability that the origin is covered in the superposition of the two processes is just the left-hand side of (2.1) and the claim follows.

Now consider the process (X, λ_1, r_1) and scale it by a factor $\alpha < 1$ to obtain a process which is equivalent in law to $(X, \alpha^{-2}\lambda_1, \alpha r_1)$. In other words, if the occurrences of (X, λ_1, r_1) are the points $\{x_1, x_2, \ldots\}$, with associated balls of radius r_1, then the occurrences of the scaled model are the points $\{\alpha x_1, \alpha x_2, \ldots\}$ with associated balls of radius αr_1. (Note that in this way we couple all processes together for $\alpha < 1$.) The CVF of $(X, \alpha^{-2}\lambda_1, \alpha r_1)$ does not depend on α. Hence it follows from (2.1) and the reasoning above that the CVF of the superposition of (X, λ_2, r_2) and $(X, \alpha^{-2}\lambda_1, \alpha r_1)$ is strictly larger than A_c. Our goal now is to show that this superposition is subcritical for α sufficiently small.

We need to review some notions from the literature. First, we define *crossing probabilities*. For $k_1, k_2 > 0$, let $\sigma((k_1, k_2), \lambda, \rho)$ be the probability —under the law $P_{\lambda,\rho}$—that the set $([0, k_1] \times [0, k_2]) \cap C$ contains a connected component which intersects both $\{0\} \times [0, k_2]$ and $\{k_1\} \times [0, k_2]$. We call such a component an LR (left–right) occupied crossing. The critical intensity corresponding to crossing probabilities is defined as

$$\lambda_S(\rho) = \inf\{\lambda; \limsup_{n \to \infty} \sigma((n, 3n), \lambda, \rho) > 0\}$$

It is shown in Menshikov[6] (see also Roy[9]) that if ρ has bounded support, then

$$\lambda_S(\rho) = \lambda_c(\rho) \tag{2.2}$$

Furthermore, the following result is a special case of Lemma 3.2 in Roy[9]:

Lemma 2.1. Consider the model (X, λ, ρ), where $0 < \rho \leqslant R$ a.s. for some $R > 0$. If for some $N \geqslant R$ and $\kappa < (25e)^{-121/4}$, we have $\sigma((N, 3N), \lambda, \rho) \leqslant \kappa$, then

$$P_{\lambda,\rho}(d(C(0)) \geqslant b) \leqslant C_1 e^{-C_2 b}$$

for all $b > 0$, where C_1 and C_2 are positive constants independent of b and where $d(\cdot)$ denotes the diameter of a set.

Remark. For arbitrary dimension d we need to have $\sigma((N, 3N, 3N,..., 3N), \lambda, \rho) \leqslant \kappa$ and $\kappa < (1/2d)(5^d e)^{-11^d}$ for the conclusion of the above lemma to be valid. Here $\sigma((N, 3N, 3N,..., 3N), \lambda, \rho)$ is the obvious notation for the crossing probability of the rectangle $[0, N] \times [0, 3N] \times \cdots \times [0, 3N]$ in the shortest direction.

All these notions were originally defined in discrete percolation. For an account of that, we refer to Kesten.[5] The results above are the continuous analogs of these results in discrete percolation.

Now we fix a $\kappa > 0$ as in Lemma 2.1. Since $\lambda_2 < \lambda_c(r_2)$, (2.2) implies that $\lambda_2 < \lambda_S(r_2)$ and we can thus find a number N so large that

$$\sigma((N, 3N), \lambda_2, r_2) < \tfrac{1}{3}\kappa$$

If there is no occupied LR crossing in $[0, N] \times [0, 3N]$, then there is a vacant TB crossing (TB stands for top to bottom) defined in the obvious way. In other words, there is at least one component in

$([0, N] \times [0, 3N]) \cap V$ intersecting the top and bottom sides of the rectangle, where V is the uncovered region as introduced earlier. We can order these components from left to right, say, and the leftmost component is called W. Only finitely many balls intersect $[0, N] \times [0, 3N]$ a.s. and hence the boundary ∂W of W has only finitely many components a.s. Hence, for n large enough, the event $E_n := \{W$ exists and all components of $\partial W \cap \operatorname{int}([0, N] \times [0, 3N])$ have distance at least n^{-1} from each other$\}$ has probability at least $1 - \frac{1}{2}\kappa$. We fix n_0 such that

$$P_{\lambda_2, r_2}(E_{n_0}) > 1 - \tfrac{1}{2}\kappa \tag{2.3}$$

Next we turn again to (X, λ_1, r_1). Since $\lambda_1 < \lambda_c(r_1)$, it follows from (2.2), Lemma 2.1, and an application of the FKG inequality that for $B_1 = [-1, 1]^2$,

$$P_{\lambda_1, r_1}(d(C(B_1)) \geqslant b) \leqslant C_3 e^{-C_4 b}$$

for all $b > 0$, where C_3 and C_4 are again positive constants independent of b. Scaling down by a factor $\alpha < 1$ yields

$$P_{\alpha^{-2}\lambda_1, \alpha r_1}(d(C(B_\alpha)) \geqslant \alpha b) \leqslant C_3 e^{-C_4 b}$$

where $B_\alpha = [-\alpha, \alpha]^2$. Taking $\alpha = m^{-1}$ for some large integer m, and $b = (2\alpha n_0)^{-1}$ [with n_0 as in (2. 3)], we obtain

$$P_{m^2\lambda_1, m^{-1}r_1}(d(C(B_{m^{-1}})) \geqslant (2n_0)^{-1}) \leqslant C_3 e^{-C_4 m/2n_0} \tag{2.4}$$

Now we combine the conclusions obtained in (2.3) and (2.4). Divide $[0, N] \times [0, 3N]$ into $3N^2m^2$ boxes with side length m^{-1}, and denote these boxes by $B^1, B^2, \ldots, B^{3N^2m^2}$. Then, from (2.4), the probability that in the model $(X, m^2\lambda_1, m^{-1}r_1)$ the event

$$F_{n_0}^m := \bigcup_{i=1}^{3N^2m^2} \{d(C(B^i)) \geqslant (2n_0)^{-1}\}$$

occurs has probability at most $3N^2m^2C_3e^{-C_4 m/2n_0}$, which tends to zero for $m \to \infty$. We now fix an m_0 such that this probability is at most $\frac{1}{3}\kappa$. If E_{n_0} occurs in (X, λ_2, r_2) and $F_{n_0}^{m_0}$ does not occur in $(X, m_0^2\lambda_1, m_0^{-1}r_1)$, then it follows that there is no occupied LR crossing in $[0, N] \times [0, 3N]$ in the superposition of the two processes. This superposition is in fact the model $(X, \lambda_2 + m_0^2\lambda_1, \rho)$, where ρ is a random variable taking values r_2 and $m_0^{-1}r_1$ with probability $\lambda_2(m_0^2\lambda_1 + \lambda_2)^{-1}$ and $m_0^2\lambda_1(m_0^2\lambda_1 + \lambda_2)^{-1}$, respectively. Hence, the probability of an occupied LR crossing of $[0, N] \times [0, 3N]$ in $(X, \lambda_2 + m_0^2\lambda_1, \rho)$ is at most $\frac{1}{2}\kappa + \frac{1}{3}\kappa < \kappa$. According to Lemma 2.1, this implies that this model is subcritical and this proves the theorem. ∎

3. PROOF OF THEOREM 1.2

If $\rho_k \Rightarrow \rho$, then the boundedness of the radii implies that also the expected volume of the balls converges. It is therefore enough to prove that $\lambda_c(\rho_k) \to \lambda_c(\rho)$, when $k \to \infty$ and this is what we shall prove.

Our strategy will be to approximate the radii by radii which take only finitely many values. Thus we first investigate the case in which both ρ_n and ρ take only finitely many values.

Lemma 3.1. Let $0 < a_1 < a_2 < \cdots < a_n < \infty$ and let ρ and ρ' be random variables taking value a_i with probability p_i and p'_i, respectively. Suppose that there exist $1 \leqslant j < l \leqslant n$ such that $p_i = p'_i$ for all $i \neq j, l$ and where p_l and p'_l are both positive. Then,

$$|\lambda_c(\rho) - \lambda_c(\rho')| \leqslant \frac{\lambda_c(a_1)}{\min\{p_l, p'_l\}} |p_j - p'_j|$$

Proof. Suppose first that $p_j > p'_j$. A simple coupling argument then shows that

$$\lambda_c(\rho) \geqslant \lambda_c(\rho') \tag{3.1}$$

Now choose $\lambda > \lambda_c(\rho')$. Consider the models $(X, \lambda l_i, a_i)$, for $i = 1,..., l-1, l+1,..., n$, where the l_i are chosen such that

$$\frac{\lambda p'_i + \lambda l_i}{\lambda(1+L)} = p_i, \qquad i = 1,..., l-1, l+1,..., n \tag{3.2}$$

where $L := l_1 + \cdots + l_{l-1} + l_{l+1} + \cdots + l_n$. Some calculations show that this boils down to the choice $l_i = p_i(p'_l/p_l) - p'_i \geqslant 0$. Next, consider the superposition of (X, λ, ρ') and $(X, \lambda l_i, a_i)$, $i = 1,..., l-1, l+1,..., n$, to obtain a model equivalent in law to $(X, \lambda(1+L), \rho)$. [To see that the radius random variable in this superposition is ρ, just use (3.2).] Since $\lambda > \lambda_c(\rho')$, the superposition is certainly supercritical. Hence

$$\lambda(1+L) > \lambda_c(\rho)$$

The above inequality holds for all $\lambda > \lambda_c(\rho')$, so we have

$$\lambda_c(\rho')(1+L) \geqslant \lambda_c(\rho)$$

From (3.2) and some elementary calculations one shows that $L = (p_l)^{-1}(p_j - p'_j)$ and the result follows, using that $\lambda_c(\rho') \leqslant \lambda_c(a_1)$. For the case $p_j < p'_j$, we just reverse the roles of ρ and ρ'. ∎

Lemma 3.2. Let $0 < a_1 < \cdots < a_n$, and let ρ be a random variable taking value a_i with probability p_i. Suppose that $p_n > 0$. For all $k = 1, 2,...$, define the random variables ρ_k taking values a_i with probability $p_{k,i}$, for all $i = 1,..., n$. If $p_{k,i} \to p_i$ for all i when $k \to \infty$, then $\lambda_c(\rho_k) \to \lambda_c(\rho)$.

Proof. We have assumed that $p_n > 0$, so we can pick $0 < \delta < p_n$. Take k_0 so large that $\sum_{i=1}^{n-1} |p_{k,i} - p_i| < \frac{1}{2}\delta$, for all $k \geqslant k_0$. Then, of course, we have $p_{k,n} > \frac{1}{2}\delta$, for all $k \geqslant k_0$. For $l = 1,..., n-1$ and $k \geqslant k_0$ let $\xi_k^{(l)}$ be the random variable defined by

$$P(\xi_k^{(l)} = a_i) = \begin{cases} p_{k,i} & \text{for } i = 1,..., l \\ p_i & \text{for } i = l+1,..., n-1 \\ p_n + \sum\limits_{i=1}^{l} (p_i - p_{k,i}) & \text{for } i = n \end{cases}$$

Clearly, $\xi_k^{(n-1)}$ has the same distribution as ρ_k and we define $\xi_k^{(0)} := \rho$.

According to Lemma 3.1, for $l = 1,..., n-1$, we have

$$|\lambda_c(\xi_k^{(l)}) - \lambda_c(\xi_k^{(l-1)})| \leqslant 2\delta^{-1} \lambda_c(a_1) \, |p_l - p_{k,l}|$$

Adding the previous inequalities over all l and using the triangle inequality, we obtain

$$|\lambda_c(\rho_k) - \lambda_c(\rho)| \leqslant 2\delta^{-1} \lambda_c(a_1) \sum_{l=1}^{n-1} |p_l - p_{k,l}|$$

for all $k \geqslant k_0$. This proves the lemma. ∎

Next we drop the assumption that p_n should be positive:

Lemma 3.3. Let ρ take values $0 < a_1 < \cdots < a_n$ with probabilities $p_1,..., p_n$, respectively. Suppose ρ_k takes values $a_1,..., a_n$ with probabilities $p_{k,1},..., p_{k,n}$. If $p_{k,i} \to p_i$ for all $1 \leqslant i \leqslant n$ then $\lambda_c(\rho_k) \to \lambda_c(\rho)$.

Proof. In view of Lemma 3.2, we need to prove this lemma for the case when there exists $1 \leqslant m \leqslant n-1$ such that

$$p_m > 0 \qquad \text{and} \qquad p_{m+1} = \cdots = p_n = 0 \tag{3.3}$$

First we show that it suffices to prove the lemma for the case $m = n-1$. Indeed, if ξ_k' and ξ_k'' take values $a_1,..., a_n$ with probabilities

$$p_{k,1}, p_{k,2},..., p_{k,m}, 0,..., 0, \sum_{i=m+1}^{n} p_{k,i}$$

and

$$p_{k,1},\, p_{k,2},\ldots,\, p_{k,m},\quad \sum_{i=m+1}^{n} p_{k_i},\, 0,\ldots,\, 0$$

respectively, then we clearly have

$$\lambda_c(\xi'_k) \leqslant \lambda_c(\rho_k) \leqslant \lambda_c(\xi''_k)$$

So it suffices to show that $\lambda_c(\rho_k)$ converges to $\lambda_c(\rho)$ when the ρ_k take at most one value larger than a_m with positive probability. Thus we henceforth assume that $m = n-1$, i.e., $p_{n-1} > 0$ and $p_n = 0$.

Next let ρ'_k be a random variable taking values $a_1,\ldots, a_n$ with probabilities $p_1, p_2,\ldots, p_{n-2}, p'_{k,n-1}, p_{k,n}$, respectively, where $p'_{k,n-1} := p_{n-1} - p_{k,n} \geqslant 0$ for k large enough, since $p_{n-1} > 0$ and $p_{k,n} \to 0$ when $k \to \infty$.

We shall now prove

$$\lim_{k\to\infty} \lambda_c(\rho'_k) = \lambda_c(\rho) \tag{3.4}$$

From our choice of ρ'_k, we observe that

$$\lambda_c(\rho'_k) \leqslant \lambda_c(\rho)$$

So to prove (3.4) we need to show that $\liminf_{k\to\infty} \lambda_c(\rho'_k) \geqslant \lambda_c(\rho)$.

Suppose there exists a λ such that $\liminf_{k\to\infty} \lambda_c(\rho'_k) < \lambda < \lambda_c(\rho)$. Since $\lambda < \lambda_c(\rho)$, by (2.2), and for κ as in Lemma 2.1, we can find an N such that

$$\sigma((N, 3N), \lambda, \rho) < \tfrac{1}{2}\kappa \tag{3.5}$$

Now we shall construct some processes so as to yield with (X, λ, ρ) a superposed model which has ρ'_k as the governing radius random variable. For this, we choose $l_{k,1},\ldots, l_{k,n-2}, l_n$ to satisfy the following relations:

$$\frac{p_i + l_{k,i}}{1 + L_k} = p_i \qquad \text{for} \quad i = 1,\ldots, n-2$$

and

$$\frac{l_{k,n}}{1 + L_k} = p_{k,n}$$

where $L_k = l_{k,1} + \cdots + l_{k,n-2} + l_{k,n}$. Since calculations show that this boils down to

$$l_{k,i} = \left(\frac{p_{n-1} - p'_{k,n-1}}{p'_{k,n-1}}\right) p_i \geqslant 0 \qquad \text{for} \quad i = 1,\ldots, n-2$$

and

$$l_{k,n} = \frac{p_{n-1}}{p'_{k,n-1}}\, p_{k,n} \geqslant 0$$

Clearly, for every $i = 1,\ldots, n-2$ and $i = n$, $l_{k,i} \to 0$ when $k \to \infty$. Thus, we can choose k large enough such that for all $i = 1,\ldots, n-2$ and $i = n$, we have

$$P_{\lambda l_{k,i}}(\text{there is at least one point of } X \text{ in } [-R, N+R] \times [-R, 3N+R]) < \frac{1}{2n}\kappa \tag{3.6}$$

where κ is as chosen before.

We define $\lambda'_k := \lambda l_{k,n} + \sum_{i=1}^{n-2} \lambda l_{k,i}$. Clearly, the superposition of the processes (X, λ, ρ), $(X, \lambda l_{k,i}, a_i)$ for all $i = 1,\ldots, n-2$ and $i = n$ is equivalent in law to the process $(X, \lambda + \lambda'_k, \rho'_k)$. For k large enough we obtain from (3.5) and (3.6) that $\sigma((N, 3N), \lambda + \lambda'_k, \rho) < \kappa$, and thus it follows from Lemma 2.1 that the superposed model is subcritical. However, by the choice of λ, (X, λ, ρ'_k) is supercritical, hence so is $(X, \lambda + \lambda'_k, \rho'_k)$, which is the desired contradiction.

Finally, to complete the proof of the lemma, we construct $\xi_k^{(l)}$ as in previous lemma, where $\xi_k^{(n-1)}$ has the same distribution as ρ_k and $\xi_k^{(0)} = \rho'_k$. This method shows that

$$|\lambda_c(\rho_k) - \lambda_c(\rho'_k)| \leqslant 2(p_{n-1})^{-1}\, \lambda_c(a_1) \sum_{i=1}^{n-2} |p_{k,i} - p_i|$$

and the lemma follows. ∎

Now we are in a position to prove Theorem 1.2. First we suppose that the supports of both ρ and ρ_k, $k = 1, 2,\ldots$, are concentrated in an interval $[a, R]$, where $a > 0$. The distribution function of ρ is denoted by F, and the distribution function of ρ_k by F_k. We can assume that both a and R are continuity points of F. Take a sequence $\{\pi_n\}$ of partitions of $[a, R]$, which we write as $\pi_n = \{a = \gamma_0^n < \gamma_1^n < \cdots < \gamma_{k_n}^n = R\}$. The partitions are chosen in such a way that π_{n+1} refines π_n, all points γ_i^n are continuity points of F, and such that $|\pi_n| := \max_{1 \leqslant i \leqslant k_n}\{\gamma_i^n - \gamma_{i-1}^n\} \to 0$, when $n \to \infty$. Now define, for all $n \geqslant 1$, the random variables $\rho^{(n)}$ and $\rho_{(n)}$ defined by the requirement that if $\rho \in (\gamma_{i-1}^n, \gamma_i^n]$, then $\rho^{(n)} = \gamma_i^n$ and $\rho_{(n)} = \gamma_{i-1}^n$. It follows from a simple coupling argument that $\lambda_c(\rho^{(n)}) \leqslant \lambda_c(\rho) \leqslant \lambda_c(\rho_{(n)}) \leqslant \lambda_c(a)$. Also, it is easy to see that $\lambda_c(\rho^{(n)})$ in increasing and $\lambda_c(\rho_{(n)})$ is decreasing in n. Now write

$$\alpha_n := \max_{1 \leqslant i \leqslant k_n} \frac{\gamma_i^n}{\gamma_{i-1}^n} \leqslant 1 + \frac{|\pi_n|}{a}$$

which tends to 1 when n tends to infinity. Hence $\rho^{(n)} \leqslant \alpha_n \rho_{(n)}$, which implies that $\lambda_c(\rho^{(n)}) \geqslant \lambda_c(\alpha_n \rho_{(n)}) = \alpha_n^{-d} \lambda_c(\rho_{(n)})$. Hence

$$\lambda_c(\rho^{(n)}) \leqslant \lambda_c(\rho) \leqslant \alpha_n^d \lambda_c(\rho^{(n)})$$

We can now write

$$\begin{aligned} \lambda_c(\rho) - \lambda_c(\rho^{(n)}) &\leqslant \left[\left(1 + \frac{|\pi_n|}{a}\right)^d - 1\right] \lambda_c(\rho^{(n)}) \\ &\leqslant \left[\left(1 + \frac{|\pi_n|}{a}\right)^d - 1\right] \lambda_c(a) =: \beta_n, \quad \text{say} \end{aligned} \tag{3.7}$$

The whole calculation can also be done for ρ_k instead of ρ and we obtain, in the obvious notation,

$$\lambda_c(\rho_k) - \lambda_c(\rho_k^{(n)}) \leqslant \beta_n \tag{3.8}$$

Now choose an $\varepsilon > 0$ and take n so large that $\beta_n < \varepsilon$. Observe that $\rho^{(n)}$ takes the value γ_i^n with probability $F(\gamma_i^n) - F(\gamma_{i-1}^n)$ and $\rho_k^{(n)}$ takes the value γ_i^n with probability $F_k(\gamma_i^n) - F_k(\gamma_{i-1}^n)$. Hence by the choice of the partitions, the fact that $\rho_k \Rightarrow \rho$, and Lemma 3.3, we see that $|\lambda_c(\rho^{(n)}) - \lambda_c(\rho_k^{(n)})| < 3\varepsilon$, for k sufficiently large. Together with (3.7) and (3.8) this proves the theorem in this case.

Next we drop the assumption that the supports are bounded from below by some positive number. Let $\delta > 0$ be a continuity point of F and let $\eta > 0$ be such that $P_{\lambda,\rho}(\rho > \delta) > \eta$. Since $\rho_k \Rightarrow \rho$, we have $P_{\lambda,\rho_k}(\rho_k > \delta) > \eta$ for k sufficiently large. Certainly, if $(X, \eta\lambda, \delta)$ is supercritical, so is (X, λ, ρ_k) and it follows that if $\eta\lambda > \lambda_c(\delta)$, then $\lambda > \lambda_c(\rho_k)$, or

$$\lambda_c(\rho_k) \leqslant \frac{1}{\eta} \lambda_c(\delta) \tag{3.9}$$

Now let $\varepsilon > 0$ and choose a to be a continuity point of F such that $F(a) < \varepsilon$, and choose k_0 so large that $F_k(a) < \varepsilon$ for all $k \geqslant k_0$. Let ρ^a be a random variable with distribution equal to the conditional distribution of ρ, given that $\rho \geqslant a$. Similarly, let ρ_a be a random variable with distribution equal to the conditional distribution of ρ given $\rho < a$. Then we have $\lambda_c(\rho^a) \leqslant \lambda_c(\rho)$.

Consider the model (X, λ, ρ^a) and $(X, \lambda l, \rho^a)$, where l is chosen such that $l(1 + l)^{-1} = P_{\lambda,\rho}(\rho \leqslant a)$. This means that

$$l = \frac{F(a)}{1 - F(a)} \tag{3.10}$$

The superposition of the two models is equivalent in law to $(X, \lambda(1 + l), \rho)$.

Thus if $\lambda > \lambda_c(\rho^a)$, then certainly this superposition is supercritical and hence $\lambda(1+l) > \lambda_c(\rho)$, i.e., $\lambda_c(\rho^a)(1+l) \geqslant \lambda_c(\rho)$. Hence

$$|\lambda_c(\rho) - \lambda_c(\rho^a)| \leqslant l\lambda_c(\rho^a) \leqslant \frac{\varepsilon}{1-\varepsilon}\lambda_c(\rho) \tag{3.11}$$

where we have used (3.10). In the same way we find, in the obvious notation and using (3.9),

$$|\lambda_c(\rho_k) - \lambda_c(\rho_k^a)| \leqslant \frac{\varepsilon}{1-\varepsilon}\lambda_c(\rho_k^a) \leqslant \frac{\varepsilon}{\eta(1-\varepsilon)}\lambda_c(\delta) \tag{3.12}$$

When $\rho_k \Rightarrow \rho$, then also $\rho_k^a \Rightarrow \rho^a$ and from the case already proved we conclude that

$$|\lambda_c(\rho_k^a) - \lambda(\rho^a)| < \varepsilon \tag{3.13}$$

for k large enough. The result now follows from combining (3.11)–(3.13). ▮

ACKNOWLEDGMENTS

R.M. would like to thank the ISI for its hospitality. The research of R.R. was carried out under a grant from his host institute. A.S. is grateful to NBHM for a doctoral fellowship.

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